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KAEHLER STRUCTURES ON $K_{\mathbf{C}}/(P,P)$

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Abstract. Let K be a compact connected semi-simple Lie group, let $G = K_{\mathbf{C}}$, and let $G = KAN$ be an Iwasawa decomposition. To a given K-invariant Kaehler structure ω on G/N, there corresponds a pre-quantum line bundle $\mathbf{L}$ on G/N. Following a suggestion of A.S. Schwarz, in a joint paper with V. Guillemin, we studied its holomorphic sections $\mathcal{O}(\mathbf{L})$ as a K-representation space. We defined a K-invariant L^2-structure on $\mathcal{O}(\mathbf{L})$, and let $H_\omega \subset \mathcal{O}(\mathbf{L})$ denote the space of square-integrable holomorphic sections. Then H_ω is a unitary K-representation space, but not all unitary irreducible K-representations occur as subrepresentations of H_ω. This paper serves as a continuation of that work, by generalizing the space considered. Let B be a Borel subgroup containing N, with commutator subgroup $(B,B) = N$. Instead of working with $G/N = G/(B,B)$, we consider $G/(P,P)$, for all parabolic subgroups P containing B. We carry out a similar construction, and recover in H_ω the unitary irreducible K-representations previously missing. As a result, we use these holomorphic sections to construct a model for K: a unitary K-representation in which every irreducible K-representation occurs with multiplicity one.

1. Introduction

Let K be a compact connected semi-simple Lie group, let $G = K_{\mathbf{C}}$ be its complexification, and let $G = KAN$ be the Iwasawa decomposition. Since G and N are complex Lie groups, G/N is a complex manifold, and G acts on G/N by left action. Let T be the centralizer of A in K, so that $H = TA$ is a Cartan subgroup of G. Since H normalizes N, there is a right action of H on G/N. We shall often be interested in the maximal compact group action of $K \times T$. We let $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \mathfrak{t}, \mathfrak{a}, \mathfrak{n}$ denote the Lie algebras of G, K, H, T, A, N respectively.

The following scheme of geometric quantization was suggested by A.S. Schwarz [12]: Equip G/N with a suitable K-invariant Kaehler structure ω, and consider the pre-quantum line bundle $\mathbf{L}$ associated to ω ([5], [11]). The Chern class of $\mathbf{L}$ is $[\omega]$, and $\mathbf{L}$ comes with a connection ∇ whose curvature is ω, as well as an invariant Hermitian structure $\langle , \rangle$. We denote by $\mathcal{O}(\mathbf{L})$ the space of holomorphic sections on $\mathbf{L}$. The K-action on G/N lifts to a K-representation on $\mathcal{O}(\mathbf{L})$. Let μ be the $K \times A$-invariant measure on G/N, which is unique up to a non-zero constant. Given a holomorphic section s of $\mathbf{L}$, we consider the integral

$$\int_{G/N} \langle s, s \rangle \mu \ .$$

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Let $H_\omega \subset \mathcal{O}(\mathbf{L})$ denote the holomorphic sections in which this integral converges. Since μ is K-invariant, H_ω becomes a unitary K-representation space. It was hoped in [12] that every irreducible K-representation occurs with multiplicity one in H_ω (called a *model* by I.M. Gelfand [7]).

By the method of highest weight, the irreducible K-representations can be labeled by the dominant integral weights in $\mathfrak{t}^*$, up to isomorphism. In joint work with V. Guillemin [4], we carried out this construction, but found that no matter how ω is chosen, the irreducibles whose highest weights lie on the wall of the Weyl chamber do not occur in the Hilbert space H_ω. Therefore, not all unitary K-irreducibles occur in H_ω. The present paper follows a suggestion of V. Guillemin ([4], p.192), by modifying the space G/N to more general classes of homogeneous spaces. As a result, we manage to recover the unitary K-irreducibles previously missing.

Let $B = HN$ be the Borel subgroup of G. Observe that its commutator subgroup is $(B, B) = N$, hence $G/N = G/(B, B)$. With this in mind, we can generalize the class of homogeneous spaces considered to $G/(P, P)$, for P a parabolic subgroup of G containing B, and (P, P) its commutator subgroup. Since P is a complex Lie group, so is (P, P); hence $G/(P, P)$ is a complex manifold. Clearly G acts on $G/(P, P)$ on the left, and we shall see that a complex subgroup of H normalizes (P, P), and hence acts on $G/(P, P)$ on the right.

Let $W \subset \mathfrak{t}^*$ denote the open Weyl chamber, and $\overline{W}$ its closure. We say that $\sigma \subset \overline{W}$ is a *cell* if there exists a subset S of the positive simple roots Δ such that

$$\sigma = \{x \in \overline{W}\ ;\ (x, S) = 0\, ,\, (x, \Delta\backslash S) > 0\}, \tag{1.1}$$

where the pairing used is the Killing form. This way, $\overline{W}$ is a disjoint union of the cells of various dimensions. Using the Killing form and the almost complex structure, it is convenient to regard the cell σ as contained in any of the spaces $\mathfrak{h}, \mathfrak{t}, \mathfrak{a}, \mathfrak{h}^*, \mathfrak{t}^*, \mathfrak{a}^*$, depending on the context. The cell σ defines a subalgebra $\mathfrak{h}_\sigma$ of $\mathfrak{h}$, by taking the complex linear span of σ. Similarly, the subalgebras $\mathfrak{t}_\sigma, \mathfrak{a}_\sigma$ are defined by intersecting $\mathfrak{h}_\sigma$ with $\mathfrak{t}, \mathfrak{a}$ respectively. These subalgebras define the subgroups $H_\sigma, T_\sigma, A_\sigma$ of H, T, A respectively. A bijective correspondence between the cells $\{\sigma\}$ and the parabolic subgroups $\{P\}$ containing B is given by the Langlands decomposition ([10], p.132)

$$P = MA_\sigma N_\sigma. \tag{1.2}$$

Fix a parabolic subgroup P containing B, with σ its corresponding cell. Since H_σ is the normalizer of (P, P) in H, it acts on $G/(P, P)$ on the right. Out of the action of the complex group $G \times H_\sigma$, we shall consider the action of the maximal compact group $K \times T_\sigma$ on $G/(P, P)$. We shall show that

Theorem I. *Let ω be a K-invariant Kaehler structure on $G/(P, P)$. Then ω is $K \times T_\sigma$-invariant if and only if it has a potential function.*

Though we shall be interested mostly in Kaehler structures, Theorem I holds also for a degenerate (1,1)-form ω. In the next theorem, we shall derive a necessary and sufficient condition for a (1,1)-form ω to be Kaehler. Let ω be a $K \times T_\sigma$-invariant (1,1)-form, so that

$$\omega = \sqrt{-1}\partial\bar{\partial}F,$$

for some function F on $G/(P, P)$. Averaging by the compact group K if necessary, we may assume that F is K-invariant. Let K^σ be the centralizer of T_σ in K. It defines a compact semi-simple subgroup K^σ_{ss} of K, given by $K^\sigma_{ss} = (K^\sigma, K^\sigma)$. We shall show that, as real manifolds and $K \times H_\sigma$-spaces,

$$G/(P,P) = (K/K^\sigma_{ss}) \times A_\sigma. \tag{1.3}$$

Therefore, the potential function F, being K-invariant, can be regarded as a function on A_σ. Since the exponential map identifies the vector space $\mathfrak{a}_\sigma$ with A_σ, F becomes a function on $\mathfrak{a}_\sigma$. The almost complex structure identifies the dual spaces $\mathfrak{a}^*_\sigma \cong \mathfrak{t}^*_\sigma$; hence the Legendre transform of F can be written as

$$L_F : \mathfrak{a}_\sigma \longrightarrow \mathfrak{t}^*_\sigma.$$

The significance of this map will become apparent shortly, when we study the moment map. We write $\log : A_\sigma \longrightarrow \mathfrak{a}_\sigma$ for the inverse of the exponential map.

The K-action on $G/(P,P)$ preserving ω is Hamiltonian: there exists a unique moment map

$$\Phi : G/(P,P) \longrightarrow \mathfrak{k}^*$$

corresponding to this action. Since Φ is K-equivariant, (1.3) implies that it is determined by its value on $A_\sigma \subset (K/K^\sigma_{ss}) \times A_\sigma$, where A_σ is imbedded as its product with the identity coset $eK^\sigma_{ss} \in K/K^\sigma_{ss}$. Meanwhile, since $\mathfrak{k}$ is semi-simple, the Killing form on $\mathfrak{k}$ is non-degenerate; which induces the inclusion $\mathfrak{t}^* \subset \mathfrak{k}^*$ from $\mathfrak{t} \subset \mathfrak{k}$.

Theorem II. *Let ω be a $K \times T_\sigma$-invariant (1,1)-form on $G/(P,P)$. Then its moment map Φ and its potential function F satisfy $\Phi(a) = \frac{1}{2} L_F(\log a) \in \mathfrak{t}^*_\sigma$ for all $a \in A_\sigma$. Further, $\omega = \sqrt{-1}\partial\bar{\partial}F$ is Kaehler if and only if:*
(i) *$F \in C^\infty(\mathfrak{a}_\sigma)$ is strictly convex; and*
(ii) *the image of $\frac{1}{2}L_F$ is contained in the cell $\sigma \subset \mathfrak{t}^*_\sigma$; i.e. $\Phi(A_\sigma) \subset \sigma$.*

Since a $K \times T_\sigma$-invariant Kaehler structure ω has a potential function F, it is exact. Therefore, it is in particular integral. Let $\mathbf{L}$ be the line bundle on $G/(P,P)$ whose Chern class is $[\omega] = 0$, equipped with a connection ∇ whose curvature is ω ([5],[11]). The topology of $\mathbf{L}$ is trivial, but the connection ∇ gives rise to interesting geometry on the holomorphic sections of $\mathbf{L}$. We recall that $\mathbf{L}$ is equipped with an invariant Hermitian structure $\langle , \rangle$. Let μ be a $K \times A_\sigma$-invariant measure on $G/(P,P)$. We consider the integral

$$\int_{G/(P,P)} \langle s, s\rangle \mu \ , \tag{1.4}$$

for holomorphic sections s of $\mathbf{L}$. As we shall see in Theorem III, convergence of this integral is determined by the image of the moment map. The $K \times T_\sigma$-action on $G/(P,P)$ lifts to a $K \times T_\sigma$-representation on $\mathcal{O}(\mathbf{L})$, the space of holomorphic sections of $\mathbf{L}$. We similarly define $H_\omega \subset \mathcal{O}(\mathbf{L})$ to be the holomorphic sections in which (1.4) converges. Since μ is K-invariant, H_ω becomes a unitary K-representation space. For a dominant integral weight λ, let $\mathcal{O}(\mathbf{L})_\lambda$ be the holomorphic sections in $\mathbf{L}$ that transform by λ under the right T_σ-action. Since the left K-action commutes with the right T_σ-action, $\mathcal{O}(\mathbf{L})_\lambda$ is a K-representation space. Let σ be the cell corresponding to the parabolic subgroup P, and let $\overline{\sigma}$ be its closure. Then

Theorem III. *The irreducible K-representation with highest weight λ occurs in $\mathcal{O}(\mathbf{L})$ if and only if $\lambda \in \overline{\sigma}$. For $\lambda \in \overline{\sigma}$, it occurs with multiplicity one, and is given by $\mathcal{O}(\mathbf{L})_\lambda$. Further, $\mathcal{O}(\mathbf{L})_\lambda$ is contained in H_ω if and only if λ lies in the image of the moment map.*

With this result, it is now clear that in [4], the singular representations are never contained in H_ω :

When $P = B$, σ becomes the open Weyl chamber W. Then Theorem II says that $\Phi(A_\sigma) \subset W$; and by K-equivariance, $\Phi(G/(P,P)) = Ad_K^*(\Phi(A_\sigma))$ does not intersect the wall $\overline{W}\backslash W$. Consequently, by Theorem III, the irreducible representations $\mathcal{O}(\mathbf{L})_\lambda$ with highest weight $\lambda \in \overline{W}\backslash W$ cannot be contained in H_ω.

Similarly, for a general parabolic subgroup P, not all $\mathcal{O}(\mathbf{L})_\lambda$ are contained in H_ω: For $\lambda \in \overline{\sigma}\backslash\sigma$, Theorems II and III say that $\mathcal{O}(\mathbf{L})_\lambda$ exists non-trivially but is not contained in H_ω.

We shall see that, however, for a suitable Kaehler structure ω on $G/(P,P)$, the image of the moment map intersects $\overline{\sigma}$ in all of σ. This way, by Theorem III, all the K-irreducibles $\mathcal{O}(\mathbf{L})_\lambda$ with highest weights $\lambda \in \sigma$ are contained in H_ω. As an application, we provide a geometric construction of a unitary K-representation, containing all the irreducibles with multiplicity one.

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2. Kaehler structures on $G/(P,P)$

The main purpose of this section is to prove Theorem I. Since K is connected and semi-simple, so is $G = K_{\mathbf{C}}$. Let P be a parabolic subgroup of G containing B, and σ the cell corresponding to P. They are related by Langlands decomposition (1.2):

$$P = MA_\sigma N_\sigma\,,$$

where A_σ is the subgroup described in §1. Then $A_\sigma \subset A, N_\sigma \subset N$, where A, N come from the Iwasawa decomposition of G. Further, A_σ normalizes N_σ, and is the centralizer of MA_σ in A. Therefore, $H_\sigma = T_\sigma A_\sigma$ is the normalizer of $(P,P) = (M,M)N_\sigma$ in H, which induces a natural right H_σ-action on $G/(P,P)$. We shall give another description of $G/(P,P)$, which reflects this right action better.

Since G is semi-simple, the Killing form is non-degenerate. Let $\mathfrak{a}_\sigma^\perp$ be the orthocomplement of $\mathfrak{a}_\sigma$ with respect to the Killing form in $\mathfrak{a}$, and $A_\sigma^\perp \subset A$ the corresponding subgroup induced by $\mathfrak{a}_\sigma^\perp$. We construct $\mathfrak{t}_\sigma^\perp, T_\sigma^\perp, \mathfrak{h}_\sigma^\perp, H_\sigma^\perp$ similarly. Let K^σ be the subgroup of K given by

$$K^\sigma = \{k \in K\ ;\ kt = tk \text{ for all } t \in T_\sigma\}.$$

Let $K^\sigma_{ss} = (K^\sigma, K^\sigma)$ be the corresponding compact semi-simple Lie group. Then

$$(K^\sigma_{ss})_{\mathbf{C}} = K^\sigma_{ss}\, A_\sigma^\perp\, (M \cap N) \tag{2.1}$$

is the Iwasawa decomposition of the complexified group $(K^\sigma_{ss})_{\mathbf{C}}$. Since $N = (M \cap N)\ N_\sigma$, it follows from (2.1) that

$$\begin{aligned} K^\sigma_{ss} A_\sigma^\perp N &= (K^\sigma_{ss})_{\mathbf{C}} N_\sigma \\ &= (K^\sigma_{\mathbf{C}})_{ss} N_\sigma \\ &= (MA_\sigma, MA_\sigma) N_\sigma \\ &= (M,M) N_\sigma \\ &= (P,P). \end{aligned} \tag{2.2}$$

Then, the Iwasawa decomposition $G = KAN$ and (2.2) imply that

$$G/(P,P) = (K/K_{ss}^{\sigma}) \times A_{\sigma}, \tag{2.3}$$

as real manifolds and $K \times H_{\sigma}$-spaces. With this description, the right action of $H_{\sigma} = T_{\sigma}A_{\sigma}$ is clear: T_{σ} acts on $(K/K_{ss}^{\sigma}) \times A_{\sigma}$ simply because it commutes with K_{ss}^{σ} and A_{σ}, while A_{σ} acts on $(K/K_{ss}^{\sigma}) \times A_{\sigma}$ by group multiplication on itself. We shall be concerned with the $K \times T_{\sigma}$-action on $G/(P,P)$.

Since $N = (B,B) \subset (P,P)$, there is a fibration

$$\pi : G/N \longrightarrow G/(P,P). \tag{2.4}$$

It follows from $G = KAN$ and (2.3) that the fiber of π is $K_{ss}^{\sigma} \times A_{\sigma}^{\perp}$. Further, π sends every right H-orbit in G/N to a right H_{σ}-orbit in $G/(P,P)$, by contracting each $H_{\sigma}^{\perp}$-coset to a point.

Given a K-invariant Kaehler structure ω on $G/(P,P)$, we want to show that it is invariant under the right T_{σ}-action if and only if it has a potential function. Our strategy is to work on the (1,1)-form $\pi^*\omega$ on G/N using results in [4], then transfer this result back to ω. Let V be the orthocomplement of $\mathfrak{t}$ in $\mathfrak{k}$ with respect to the Killing form, so that $\mathfrak{k} = \mathfrak{t} \oplus V$. The Killing form also induces $\mathfrak{t}^* \subset \mathfrak{k}^*$ from $\mathfrak{t} \subset \mathfrak{k}$. If F is a function on A, then by the exponential map, it becomes a function on $\mathfrak{a}$. Using the almost complex structure, $\mathfrak{a}^* \cong \mathfrak{t}^*$. Therefore, the Legendre transform of F can be written as

$$L_F : \mathfrak{a} \longrightarrow \mathfrak{t}^*. \tag{2.5}$$

Given $\xi \in \mathfrak{k}$, we let $\xi^\sharp$ denote its infinitesimal vector field on G/N induced by the K-action. Let J be the almost complex structure on G/N. For $\eta = J\xi \in \mathfrak{a}$, where $\xi \in \mathfrak{t}$, we define $\eta^\sharp$ to be the vector field $J\xi^\sharp$. Let $a \in A \subset KA = G/N$. Then its tangent space is $T_a(G/N) = \mathfrak{h}_a^\sharp \oplus V_a^\sharp$. We recall the following result from [4]:

Proposition 2.1 ([4])**.** *Let ω be a $K \times T$-invariant (1,1)-form on G/N. Then $\omega = \sqrt{-1}\partial\bar{\partial}F$, where $F \in C^\infty(A)$ by K-invariance. It satisfies $\omega(\mathfrak{h}^\sharp, V^\sharp)_a = 0$. The K-action is Hamiltonian, with moment map $\Phi : G/N \longrightarrow \mathfrak{k}^*$ satisfying*

(i) $\Phi(a) \in \mathfrak{t}^*$ *for all* $a \in A \subset KA = G/N$;

(ii) $\Phi : A \longrightarrow \mathfrak{t}^*$ *is given by* $\Phi(a) = \frac{1}{2}L_F(\log a)$.

Let $m = \dim\sigma, n = \dim\mathfrak{t}$. Let $\{\lambda_1, ..., \lambda_r\}$ be the positive roots of $\mathfrak{g}$, where $\{\lambda_1, ..., \lambda_n\}$ are simple. Here $m \le n \le r$. Then $\dim V = 2r$, and $\dim\mathfrak{k} = n + 2r$. In the following proposition, we give a useful decomposition of V. Recall that we defined the cell σ in (1.1) using a subset S of the positive simple roots Δ. By switching the roles of S and $\Delta\backslash S$, we can define another cell σ', with dimension $n - m$. We call σ' the complementary cell to σ. Let J be the almost complex structure on $\mathfrak{k} \oplus \mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. Recall that V is the orthocomplement of $\mathfrak{t}$ in $\mathfrak{k}$.

Proposition 2.2. *Let σ, σ' be complementary cells of dimensions $m, n-m$ respectively, where $m \le n \le r = \frac{1}{2}\dim V$. There exists a decomposition $V = \bigoplus_1^r V_i$ into two-dimensional subspaces V_i. Each V_i is preserved by J and satisfies $[V_i, V_i] \subset \mathfrak{t}$. Further,*

(i) $\mathfrak{t}_{\sigma'}^{\perp} = \bigoplus_1^m [V_i, V_i]$,

(ii) $\mathfrak{t}_{\sigma}^{\perp} = \bigoplus_{m+1}^n [V_i, V_i]$.

If ω is a $K \times T$-invariant (1,1)-form on G/N, then $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for all $i \ne j, a \in A \subset KA = G/N$.

Proof. Let $\{\lambda_1, ..., \lambda_r\}$ be the positive roots of $\mathfrak{g}$, indexed such that the first n of them are simple. Further, we can require that

$$(\lambda_i, \sigma) > 0 \ , \ (\lambda_i, \sigma') = 0 \ \ ; \ \ i = 1, ..., m,$$

and

$$(\lambda_i, \sigma) = 0 \ , \ (\lambda_i, \sigma') > 0 \ \ ; \ \ i = m+1, ..., n,$$

where the pairing is the Killing form.

Let $\mathfrak{g}_{\pm i}$ be the root spaces corresponding to $\pm\lambda_i$. Then there exist $\xi_{\pm i} \in \mathfrak{g}_{\pm i}$ such that

$$\{ \ \zeta_i = \xi_i - \xi_{-i} \quad , \quad \gamma_i = \sqrt{-1}(\xi_i + \xi_{-i}) \ \}_{i=1,...,r} \tag{2.6}$$

form a basis of V ([8], p.421). Here $\{\zeta_i, \gamma_i\}$ are orthogonal to $\mathfrak{t}$ because the root spaces $\mathfrak{g}_i$ are orthogonal to $\mathfrak{h}$. Further, $\{\xi_{\pm i}\}$ can be chosen such that $[\zeta_i, \gamma_i] \in \mathfrak{t}$, and dual to $\lambda_i \in \mathfrak{t}^*$ with respect to the Killing form. We define

$$V_i = \mathbf{R}(\zeta_i, \gamma_i).$$

Then $[V_i, V_i] \subset \mathfrak{t}$. Let J be the almost complex structure on $\mathfrak{k} \oplus \mathfrak{a} = \mathfrak{g}/\mathfrak{n}$. From (2.6), it follows that J sends ζ_i to γ_i, and sends γ_i to $-\zeta_i$. Therefore, each V_i is preserved by J.

For $i = 1, ..., m$, $(\lambda_i, \sigma') = 0$. Since $[\zeta_i, \gamma_i]$ is dual to λ_i, it follows that $[\zeta_i, \gamma_i] \in \mathfrak{t}^\perp_{\sigma'}$. Hence $[V_i, V_i] \subset \mathfrak{t}^\perp_{\sigma'}$ for $i = 1, ..., m$. But the dual vectors of $\lambda_1, ..., \lambda_m$ form a basis of $\mathfrak{t}^\perp_{\sigma'}$; hence $\mathfrak{t}^\perp_{\sigma'} = \bigoplus_1^m [V_i, V_i]$.

For $i = m+1, ..., n$, $(\lambda_i, \sigma) = 0$. By a similar argument, $\mathfrak{t}^\perp_\sigma = \bigoplus_{m+1}^n [V_i, V_i]$.

Let ω be a $K \times T$-invariant (1,1)-form on G/N. Suppose that $i \neq j$; we want to show that $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for $a \in A \subset KA = G/N$. Let $p : \mathfrak{k} \longrightarrow \mathfrak{t}$ be the orthogonal projection, annihilating V. Let $\xi \in V_i, \eta \in V_j$. From (2.6), it follows that $[\xi, \eta]$ is either 0 or in V_k, depending on whether $\lambda_i + \lambda_j$ is some positive root λ_k. In any case,

$$p[\xi, \eta] = 0 \ \ ; \ \ \xi \in V_i, \eta \in V_j. \tag{2.7}$$

Let $\Phi : G/N \longrightarrow \mathfrak{k}^*$ be the moment map corresponding to the K-action preserving ω. Then $\Phi(a) \in \mathfrak{t}^*$, by Proposition 2.1. Consequently,

$$\begin{aligned} \omega(\xi^\sharp, \eta^\sharp)_a &= (\Phi(a), [\xi, \eta]) \\ &= (\Phi(a), p[\xi, \eta]) \quad \text{since } \Phi(a) \in \mathfrak{t}^* \\ &= 0. \end{aligned}$$

Therefore, $\omega(V_i^\sharp, V_j^\sharp)_a = 0$ for $i \neq j$. This proves the proposition. □

Let ω be a $K \times T_\sigma$-invariant Kaehler structure on $G/(P,P)$. Let π be the fibration in (2.4). Then $\pi^*\omega$ is a $K \times TA^\perp_\sigma$-invariant (1,1)-form on G/N. By Proposition 2.1, it has the form

$$\pi^*\omega = \sqrt{-1}\partial\bar{\partial} f,$$

where f is a K-invariant function on G/N. Since $G/N = KA$, $f \in C^\infty(A)$. We shall show that f can be replaced with another function F which is in the image of

$$\pi^* : C^\infty(G/(P,P)) \longrightarrow C^\infty(G/N),$$

so that we get a potential function for ω.

Let σ be the cell which corresponds to P by (1.2), and σ' its complementary cell. Then σ' defines subgroups $H_{\sigma'}, T_{\sigma'}, A_{\sigma'}$ of H, T, A respectively. By taking the orthocomplements of the Lie algebras $\mathfrak{h}_{\sigma'}, \mathfrak{t}_{\sigma'}, \mathfrak{a}_{\sigma'}$, we construct the subgroups $H^{\perp}_{\sigma'}, T^{\perp}_{\sigma'}, A^{\perp}_{\sigma'}$ as before. Note in particular that $A = A^{\perp}_{\sigma} A^{\perp}_{\sigma'}$. Define $F \in C^{\infty}(A)$ by

$$F = \rho^* f \quad , \quad \rho : A \longrightarrow A^{\perp}_{\sigma'} \longrightarrow A, \tag{2.8}$$

where ρ is the composite function of the submersion $A \longrightarrow A^{\perp}_{\sigma'}$ annihilating $A^{\perp}_{\sigma}$, followed by the inclusion $A^{\perp}_{\sigma'} \longrightarrow A$. By $G/N = KA$, F extends uniquely to a $K \times TA^{\perp}_{\sigma}$-invariant function on G/N. Note that F is in the image of π^*. We define the $K \times TA^{\perp}_{\sigma}$-invariant (1,1)-form

$$\Omega = \sqrt{-1}\partial\bar{\partial}F.$$

We shall show that

$$\Omega = \pi^*\omega. \tag{2.9}$$

Here both Ω and $\pi^*\omega$ are $K \times TA^{\perp}_{\sigma}$-invariant. Since $G/N = KA^{\perp}_{\sigma'}A^{\perp}_{\sigma}$, we only have to compare them at $a \in A^{\perp}_{\sigma'}$. Also, Proposition 2.1 says that $\mathfrak{h}^{\sharp}_a$ and $V^{\sharp}_a$ are complementary with respect to both Ω_a and $\pi^*\omega_a$. Therefore, (2.9) will follow if we can show that

$$\Omega(\xi^{\sharp}, \eta^{\sharp})_a = \pi^*\omega(\xi^{\sharp}, \eta^{\sharp})_a \;\; ; \;\; \xi, \eta \in \mathfrak{h} \text{ or } \xi, \eta \in V \, , \, a \in A^{\perp}_{\sigma'}. \tag{2.10}$$

This will be checked by the following two lemmas. Recall that $L_F, L_f : \mathfrak{a} \longrightarrow \mathfrak{t}^*$ are the Legendre transforms of F and f, described in (2.5).

Lemma 2.3. *$\Omega(\xi^{\sharp}, \eta^{\sharp})_a = \pi^*\omega(\xi^{\sharp}, \eta^{\sharp})_a$ for all $\xi, \eta \in V, a \in A^{\perp}_{\sigma'}$.*

Proof. By Proposition 2.2, the spaces $(V_1)^{\sharp}_a, ..., (V_r)^{\sharp}_a$ are pairwise complementary with respect to Ω_a and $\pi^*\omega_a$, $a \in A^{\perp}_{\sigma'}$. Therefore, to prove this lemma, we may consider $\xi, \eta \in V_i$ for each component V_i seperately. Since each V_i is two-dimensional, it suffices to consider $\xi = \zeta_i, \eta = \gamma_i$. Let

$$\Phi_F, \Phi_f : G/N \longrightarrow \mathfrak{k}^*$$

be the moment maps of the K-actions preserving $\Omega, \pi^*\omega$ respectively. We recall from Proposition 2.1 that $\Phi_F(a) = \frac{1}{2}L_F(\log a), \Phi_f(a) = \frac{1}{2}L_f(\log a)$. We follow the indices $i = 1, ..., r$ used in Proposition 2.2, as well as the cells σ, σ' of dimensions $m, n-m$ respectively. In what follows, we break up our arguments into three cases, according to the different values of the index i.

Case 1. $i = 1, ..., m$.

$$\begin{aligned} \Omega(\zeta^{\sharp}_i, \gamma^{\sharp}_i)_a &= (\Phi_F(a), [\zeta_i, \gamma_i]) \\ &= (\tfrac{1}{2}L_F(\log a), [\zeta_i, \gamma_i]). \end{aligned}$$

By Proposition 2.2, $[\zeta_i, \gamma_i] \in \mathfrak{t}^{\perp}_{\sigma'}$, for $i = 1, ..., m$. By (2.8), $L_F(\log a)$ and $L_f(\log a)$ agree on $\mathfrak{t}^{\perp}_{\sigma'}$, for $a \in A^{\perp}_{\sigma'}$. Therefore, the last expression is

$$\begin{aligned} (\tfrac{1}{2}L_f(\log a), [\zeta_i, \gamma_i]) &= (\Phi_f(a), [\zeta_i, \gamma_i]) \\ &= \pi^*\omega(\zeta^{\sharp}_i, \gamma^{\sharp}_i)_a. \end{aligned}$$

Case 2. $i = m+1, ..., n$.

We recall (2.6), which implies that

$$[v, \zeta_i] = \sqrt{-1}(\lambda_i, v)\gamma_i \quad , \quad [v, \gamma_i] = -\sqrt{-1}(\lambda_i, v)\zeta_i \tag{2.11}$$

for all $v \in \mathfrak{t}$. Therefore, the Lie algebra $\mathfrak{k}^{\sigma}$ of K^{σ} is given by

$$\mathfrak{k}^{\sigma} \; = \; \{\xi \in \mathfrak{k} \; ; \; [\xi, \sigma] = 0\} \; = \; \mathfrak{t} \oplus_{(\lambda_i, \sigma)=0} V_i.$$

The center of this Lie algebra is $\mathfrak{t}_\sigma$; hence the semi-simple Lie algebra $\mathfrak{k}^\sigma_{ss}$ is given by

$$\mathfrak{k}^\sigma_{ss} \;=\; \mathfrak{t}^\perp_\sigma \oplus_{(\lambda_i,\sigma)=0} V_i. \tag{2.12}$$

For $i = m+1, ..., n$, $(\lambda_i, \sigma) = 0$; hence $\zeta_i, \gamma_i \in \mathfrak{k}^\sigma_{ss}$. But K^σ_{ss} is in the fiber of π, so $\imath(\xi^\sharp)\pi^*\omega_a = 0$ for all $\xi \in V_i$.

We shall show that

$$\imath(\xi^\sharp)\Omega_a = 0$$

for all $\xi \in V_i$. Since each V_i is two-dimensional, this will follow if we can show that $\Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a = 0$, for $i = m+1, ..., n$. But

$$\Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a = (\frac{1}{2} L_F(\log a), [\zeta_i, \gamma_i]) = 0,$$

since $[\zeta_i, \gamma_i] \in \mathfrak{t}^\perp_\sigma$ and by (2.8), $L_F(\log a)$ vanishes there.

Case 3. $i = n+1, ..., r$.

From Cases 1, 2, we see that $L_F(\log a), L_f(\log a) \in \mathfrak{t}^*$ agree on the spaces $\mathfrak{t}^\perp_\sigma, \mathfrak{t}^\perp_{\sigma'}$. Since $\mathfrak{t} = \mathfrak{t}^\perp_\sigma \oplus \mathfrak{t}^\perp_{\sigma'}$, it follows that $L_F(\log a) = L_f(\log a) \in \mathfrak{t}^*$. Therefore,

$$\begin{aligned}\Omega(\zeta_i^\sharp, \gamma_i^\sharp)_a &= (\Phi_F(a), [\zeta_i, \gamma_i]) \\ &= (\frac{1}{2} L_F(\log a), [\zeta_i, \gamma_i]) \\ &= (\frac{1}{2} L_f(\log a), [\zeta_i, \gamma_i]) \\ &= (\Phi_f(a), [\zeta_i, \gamma_i]) \\ &= \pi^*\omega(\zeta_i^\sharp, \gamma_i^\sharp)_a.\end{aligned}$$

This proves Lemma 2.3. □

Lemma 2.4. $\Omega(\xi^\sharp, \eta^\sharp)_a = \pi^*\omega(\xi^\sharp, \eta^\sharp)_a$ *for all* $\xi, \eta \in \mathfrak{h}, a \in A^\perp_{\sigma'}$.

Proof. Let $\mathfrak{h}_\sigma, \mathfrak{h}_{\sigma'}$ denote the subalgebras of $\mathfrak{h}$, by taking the complex linear spans of σ, σ' respectively. Let $\mathfrak{h}^\perp_\sigma, \mathfrak{h}^\perp_{\sigma'}$ denote their orthocomplements with respect to the Killing form. Then $\mathfrak{h} = \mathfrak{h}^\perp_\sigma \oplus \mathfrak{h}^\perp_{\sigma'}$.

Case 1. $\xi, \eta \in \mathfrak{h}^\perp_{\sigma'}$.

Let $\iota : H^\perp_{\sigma'} \longrightarrow H$ denote the inclusion. From (2.8), we get

$$\sqrt{-1}\partial\bar\partial(\iota^* F) = \sqrt{-1}\partial\bar\partial(\iota^* f),$$

where $\partial, \bar\partial$ are Dolbeault operators on $H^\perp_{\sigma'}$ here. Therefore, given $a \in A^\perp_{\sigma'} \subset H^\perp_{\sigma'}$,

$$\begin{aligned}\Omega(\xi^\sharp, \eta^\sharp)_a &= (\iota^*\Omega)(\xi^\sharp, \eta^\sharp)_a \\ &= (\sqrt{-1}\partial\bar\partial(\iota^* F))(\xi^\sharp, \eta^\sharp)_a \\ &= (\sqrt{-1}\partial\bar\partial(\iota^* f))(\xi^\sharp, \eta^\sharp)_a \\ &= (\iota^*\pi^*\omega)(\xi^\sharp, \eta^\sharp)_a \\ &= \pi^*\omega(\xi^\sharp, \eta^\sharp)_a.\end{aligned}$$

Case 2. $\xi \in \mathfrak{h}^\perp_\sigma$.

We shall show that

$$\imath(\xi^\sharp)\pi^*\omega_a \;=\; \imath(\xi^\sharp)\Omega_a \;=\; 0, \tag{2.13}$$

which completes the proof of this lemma. Since $\pi^*\omega$ and Ω are *(1,1)*-forms, it suffices to check (2.13) for $\xi \in \mathfrak{t}^\perp_\sigma$.

The fiber of π is $K_{ss}^{\sigma} \times A_{\sigma}^{\perp}$, which contains $H_{\sigma}^{\perp}$. Therefore,

$$\imath(\xi^{\sharp})\pi^{*}\omega_a = 0.$$

We observe that, as complex manifolds,

$$H = \mathbf{C}^n/\mathbf{Z}^n, \quad H_{\sigma}^{\perp} = \mathbf{C}^{n-m}/\mathbf{Z}^{n-m}, \quad H_{\sigma'}^{\perp} = \mathbf{C}^m/\mathbf{Z}^m,$$

and $H = H_{\sigma}^{\perp} H_{\sigma'}^{\perp}$. We introduce complex coordinates $\{z_1, ..., z_m\}$ on $H_{\sigma'}^{\perp}$ as well as $\{z_{m+1}, ..., z_n\}$ on $H_{\sigma}^{\perp}$, so that H adopts the product coordinates. Let $z = x + \sqrt{-1}y$, and let x, y be the coordinates on T, A respectively. From $H = TA, G/N = KA$ and $T \subset K$, we get a natural holomorphic imbedding $\iota : H \longrightarrow G/N$. Then $\iota^{*}F$, being T-invariant, is a function on y only. For simplicity we still denote it by F. It follows from (2.8) that

$$\frac{\partial F}{\partial y_i} = 0 \text{ for } i = m+1, ..., n.$$

Therefore, for $a \in A_{\sigma'}^{\perp}$,

$$\begin{aligned} \imath(\xi^{\sharp})(\iota^{*}\Omega)_a &= \imath(\xi^{\sharp})(\sqrt{-1}\partial\bar{\partial}F)_a \\ &= \imath(\xi^{\sharp}) \left(\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k \right) \\ &= \imath(\xi^{\sharp}) \left(\frac{1}{2} \sum_{j,k=1}^{m} \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k \right). \end{aligned} \tag{2.14}$$

On the other hand, since $\xi \in \mathfrak{t}_{\sigma}^{\perp}$, the vector field $\xi^{\sharp}$ on H is of the form

$$\xi^{\sharp} = \sum_{m+1}^{n} c_i \frac{\partial}{\partial x_i}.$$

This, together with (2.14), implies that

$$\imath(\xi^{\sharp})\Omega_a = 0.$$

This proves (2.13).

Combining the results in Cases 1,2, we have proved Lemma 2.4. □

Lemmas 2.3 and 2.4 imply (2.10), and hence (2.9). Namely, we have shown that given a $K \times T_{\sigma}$-invariant Kaehler structure ω on $G/(P,P)$, there exists a function F, which is in the image of π^{*} by virtue of (2.8), such that

$$\pi^{*}\omega = \sqrt{-1}\partial\bar{\partial}F.$$

Since F is in the image of π^{*}, and since π^{*} is injective, it follows that ω itself has a potential function.

Conversely, suppose that a K-invariant Kaehler structure ω on $G/(P,P)$ has a potential function F. Averaging by the compact group K if necessary, we may assume that F is K-invariant. But by (2.3), this means that F is just a function on A_{σ}, and is automatically $K \times T_{\sigma}$-invariant. Then ω is also $K \times T_{\sigma}$-invariant. This proves Theorem I.

We note that our arguments do not require ω to be positive definite. Namely, Theorem I holds even if ω is merely a K-invariant $(1,1)$-form. In the next section, we use the moment map to derive a necessary and sufficient condition for a $K\times T_\sigma$-invariant $(1,1)$-form to be Kaehler.

3. Moment map

Let ω be a $K\times T_\sigma$-invariant $(1,1)$-form on $G/(P,P)$, with moment map

$$\Phi : G/(P,P) \longrightarrow \mathfrak{k}^*$$

corresponding to the Hamiltonian action of K on $G/(P,P)$ preserving ω. It is easy to see that this action is Hamiltonian; either from the semi-simplicity of K ([6], §26), or from the fact that $\omega = \sqrt{-1}\partial\bar\partial F$ implies $\omega = d\beta$ for some K-invariant real 1-form β ([1], Theorem 4.2.10). We shall study the moment map Φ, and derive a necessary and sufficient condition for ω to be Kaehler.

Suppose now that ω is a $K\times T_\sigma$-invariant Kaehler structure. We want to derive the two conditions stated in Theorem II. By Theorem I, ω has a potential function F. Averaging by K if necessary, we may assume that F is K-invariant. By (2.3), $G/(P,P) = (K/K^\sigma_{ss})\times A_\sigma$; so the K-invariant function F is just a function on A_σ. Let π be the fibration in (2.4). Then

$$\Phi\circ\pi : G/N \longrightarrow \mathfrak{k}^*$$

is the moment map corresponding to the K-action on $(G/N, \pi^*\omega)$. Recall that P corresponds to a cell σ via (1.2). Also, $G/N = KA$ and $G/(P,P) = (K/K^\sigma_{ss})\times A_\sigma$ induce the inclusions

$$A \hookrightarrow \{e\}\times A \subset KA = G/N, \quad A_\sigma \hookrightarrow \{eK^\sigma_{ss}\}\times A_\sigma \subset (K/K^\sigma_{ss})\times A_\sigma = G/(P,P)\ .$$

Therefore, we can regard A and A_σ as contained in G/N and $G/(P,P)$ respectively. Note that $\pi(A) = A_\sigma$. From Proposition 2.1, we see that

$$(\Phi\circ\pi)(A) \subset \mathfrak{t}^*.$$

Since the fibration π sends A to A_σ, it follows that $\Phi(A_\sigma)\subset\mathfrak{t}^*$. By K-equivariance of Φ, $\Phi|_{A_\sigma}$ determines Φ entirely. The exponential map from $\mathfrak{a}_\sigma$ to A_σ is a diffeomorphism, and we let log be its inverse. This way, the potential function F becomes a function on $\mathfrak{a}_\sigma$. Then, by the almost complex structure, $\mathfrak{a}^*_\sigma \cong \mathfrak{t}^*_\sigma$. Consequently, the Legendre transform of F is

$$L_F : \mathfrak{a}_\sigma \longrightarrow \mathfrak{t}^*_\sigma.$$

We shall show that

$$\Phi : A_\sigma \longrightarrow \mathfrak{t}^*$$

is given by $\Phi(a) = \frac{1}{2}L_F(\log a)$ for all $a\in A_\sigma$. Let

$$\imath : H_\sigma \longrightarrow G/(P,P)$$

be the natural holomorphic imbedding of $H_\sigma = T_\sigma A_\sigma$. Then $\imath^*\omega$ is a T_σ-invariant Kaehler structure on $T_\sigma A_\sigma$, with potential function $\imath^*F$. For simplicity, we still write $\imath^*F$ as F. Let m be the dimension of the cell σ. Then, as a complex manifold, $H_\sigma = \mathbf{C}^m/\mathbf{Z}^m$. Therefore, we can introduce complex coordinates $\{z_1,...,z_m\}$ on H_σ, where

$$\begin{gathered} H_\sigma = \mathbf{C}^m/\mathbf{Z}^m = \{z_1,...,z_m\}\ ,\ T_\sigma = \mathbf{R}^m/\mathbf{Z}^m = \{x_1,...,x_m\}\ , \\ A_\sigma = \mathbf{R}^m = \{y_1,...,y_m\}\ ,\ z_i = x_i + \sqrt{-1}y_i\ . \end{gathered} \tag{3.1}$$

Since F is T_σ-invariant, it is a function on y only. Then $\imath^*\omega$ becomes (here $\partial, \bar\partial$ are Dolbeault operators on H_σ)

$$\imath^*\omega = \sqrt{-1}\partial\bar\partial F = \frac{1}{2}\sum_{j,k=1}^{m} \frac{\partial^2 F}{\partial y_j \partial y_k} dx_j \wedge dy_k, \tag{3.2}$$

where $F \in C^\infty(\mathbf{R}^m)$. Since ω is Kaehler, so is $\imath^*\omega$; and (3.2) says that $\imath^*\omega$ is Kaehler if and only if the Hessian matrix of F is positive definite, i.e. F is strictly convex.

The moment map Φ of the K-action on $(G/(P,P),\omega)$ restricts to be the moment map Φ' of the T_σ-action on $(T_\sigma A_\sigma, \imath^*\omega)$. Let

$$\beta = -\frac{1}{2}\sum_{j=1}^{m} \frac{\partial F}{\partial y_j} dx_j$$

be a T_σ-invariant 1-form on $T_\sigma A_\sigma$. From (3.2), it follows that $d\beta = \imath^*\omega$, so the moment map Φ' of the T_σ-action is

$$\begin{aligned}(\Phi'(ta),\xi) &= -(\beta, \xi^\sharp)(ta)\\ &= \left(\frac{1}{2}\sum_{j=1}^{m}\frac{\partial F}{\partial y_j}dx_j, \sum_{k=1}^{m}\xi_k \frac{\partial}{\partial x_k}\right)(ta)\\ &= \frac{1}{2}\sum_{j=1}^{m}\frac{\partial F}{\partial y_j}(a)\xi_j\\ &= \frac{1}{2}(L_F(a),\xi)\ ,\end{aligned}$$

where $ta \in T_\sigma A_\sigma, \xi \in \mathfrak{t} = \mathbf{R}^m$. Our computation identifies $\mathfrak{a}$ with A by the exponential map, so in fact $\Phi'(ta) = \frac{1}{2}L_F(\log a)$ for all $ta \in T_\sigma A_\sigma$. But Φ and Φ' agree on A_σ, so $\Phi(a) = \frac{1}{2}L_F(\log a)$. Hence $\Phi(A_\sigma) \subset \mathfrak{t}_\sigma^*$. We claim further that $\Phi(A_\sigma) \subset \sigma$:

Let $V_i \subset V \subset \mathfrak{k}$ be the subspaces constructed in Proposition 2.2, and let $\{\zeta_i, \gamma_i\} \in V_i$ be the vectors in (2.6). Recall that these indices are with respect to the positive roots $\{\lambda_i\}$. Since $G/(P,P) = (K/K_{ss}^\sigma) \times A_\sigma$, the infinitesimal vector fields $\zeta_i^\sharp, \gamma_i^\sharp$ on $G/(P,P)$ are non-zero if and only if $\zeta_i, \gamma_i \notin \mathfrak{k}_{ss}^\sigma$. By (2.12), this is equivalent to $(\lambda_i,\sigma) > 0$. Let J be the almost complex structure in $G/(P,P)$, $a \in A_\sigma$, and $(\lambda_i,\sigma) > 0$, so that $\zeta_i^\sharp, \gamma_i^\sharp \neq 0$. By (2.6), $J\zeta_i = \gamma_i$. Since ω is Kaehler,

$$\begin{aligned} 0 &< \omega(\zeta_i^\sharp, J\zeta_i^\sharp)_a\\ &= \omega(\zeta_i^\sharp, \gamma_i^\sharp)_a\\ &= (\Phi(a), [\zeta_i,\gamma_i])\\ &= (\Phi(a), \lambda_i).\end{aligned} \tag{3.3}$$

We have shown that, for all $a \in A_\sigma$, $(\Phi(a),\lambda_i) > 0$ whenever λ_i is a positive root satisfying $(\lambda_i,\sigma) > 0$. This, together with $\Phi(A_\sigma) \subset \mathfrak{t}_\sigma^*$, implies that $\Phi(A_\sigma) \subset \sigma$, as claimed.

We have shown that if ω is Kaehler, then the two conditions stated in Theorem II have to be satisfied. We next show that, conversely, these two conditions are sufficient for ω to be Kaehler.

Recall that the infinitesimal vector field $\xi^\sharp$ on $G/(P,P)$ vanishes if $\xi \in \mathfrak{k}_{ss}^\sigma$. Hence the tangent space at $a \in A_\sigma \subset G/(P,P)$ is spanned by $(\mathfrak{k}_{ss}^{\sigma\perp})_a^\sharp, (\mathfrak{a}_\sigma)_a^\sharp$. Here we define

$\eta^\sharp$ for $\eta = J\xi \in \mathfrak{a}_\sigma$ by $\eta^\sharp = J\xi^\sharp$, where $\xi \in \mathfrak{t}_\sigma$. However, it follows from (2.12) that

$$\mathfrak{k}_{ss}^{\sigma\perp} \;=\; \mathfrak{t}_\sigma \oplus_{(\lambda_i,\sigma)>0} V_i,$$

where V_i is the space described in Proposition 2.2. Here the distinct V_i are orthogonal to one another, due to the orthogonality of the root spaces $\mathfrak{g}_i$ ([8], p.166). Consequently, the tangent space at $a \in A_\sigma \subset G/(P,P)$ is

$$T_a(G/(P,P)) = (\mathfrak{h}_\sigma)^\sharp_a \oplus_{(\lambda_i,\sigma)>0} (V_i)^\sharp_a. \tag{3.4}$$

We claim that $\omega(\mathfrak{h}^\sharp_\sigma, V^\sharp_i)_a = \omega(V^\sharp_i, V^\sharp_j)_a = 0$, for $i \neq j$.

Since J preserves $\mathfrak{h}_\sigma$ and V_i, and ω is a $(1,1)$-form, the first part follows if we can show that $\omega(\mathfrak{t}^\sharp_\sigma, V^\sharp_i)_a = 0$. Let $p : \mathfrak{k} \longrightarrow \mathfrak{t}$ be the orthogonal projection annihilating V. Let $\xi \in \mathfrak{t}_\sigma, \eta \in V_i$. Then $p[\xi,\eta] = 0$, by (2.11). Since $\Phi(a) \in \mathfrak{t}^*$ for $a \in A$,

$$\omega(\xi^\sharp, \eta^\sharp)_a \;=\; (\Phi(a), [\xi,\eta]) \;=\; (\Phi(a), p[\xi,\eta]) \;=\; 0.$$

Hence $\omega(\mathfrak{h}^\sharp_\sigma, V^\sharp_i)_a = 0$. For $i \neq j$, it follows from (2.7) that $p[V_i, V_j] = 0$. So, by a similar argument, $\omega(V^\sharp_i, V^\sharp_j)_a = 0$, as claimed.

Therefore, by K-invariance of ω and (3.4), the positive definiteness of ω follows if we can check that

$$\omega(\xi^\sharp, J\xi^\sharp)_a > 0 \;\;;\;\; \xi \in \mathfrak{h}_\sigma \text{ or } \xi \in V_i\,,\, (\lambda_i,\sigma) > 0, a \in A_\sigma. \tag{3.5}$$

But they follow from the two conditions of Theorem II: Condition (i) of Theorem II implies that the expression in (3.2) is positive definite and hence (3.5) holds for $\xi \in \mathfrak{h}_\sigma$. Condition (ii) of Theorem II implies that $(\Phi(a), \lambda_i) > 0$ whenever $(\lambda_i, \sigma) > 0$, so it follows from (3.3) that (3.5) holds for $\xi \in V_i$. This proves Theorem II.

4. Line bundle

Fix a $K \times T_\sigma$-invariant Kaehler structure ω on $G/(P,P)$. By Theorem I, ω has a potential function F. Recall that P determines the subgroup A_σ by (1.2). By K-invariance and (2.3), we can regard F as a function on A_σ. In particular, the expression $\omega = \sqrt{-1}\partial\bar{\partial}F$ also implies that ω is exact. Hence ω is integral, and there exists a complex line bundle $\mathbf{L}$ on $G/(P,P)$ whose Chern class is $[\omega] = 0$, equipped with a connection ∇ whose curvature is ω, as well as an invariant Hermitian structure $\langle,\rangle$ ([5], [11]). The line bundle $\mathbf{L}$ is trivial since $[\omega] = 0$, but the connection ∇ gives rise to interesting geometry. We say that a section s is holomorphic if ∇s annihilates anti-holomorphic vector fields on $G/(P,P)$. We shall show that the $K \times T_\sigma$-action on $G/(P,P)$ lifts to a $K \times T_\sigma$-representation on the space of holomorphic sections of $\mathbf{L}$. To do this, we shall construct a global trivialization of $\mathbf{L}$. The following topological property of $G/(P,P)$ is useful in this construction:

Lemma 4.1. $H^1(G/(P,P), \mathbf{C}) = 0$.

Proof. By (2.3), $G/(P,P) = (K/K^\sigma_{ss}) \times A_\sigma$. Since A_σ is Euclidean, it suffices to show that $H^1(K/K^\sigma_{ss}, \mathbf{C}) = 0$.

The fibration $K \longrightarrow K/K^\sigma_{ss}$ induces a long exact sequence of homotopy groups,

$$... \longrightarrow \pi_1(K) \longrightarrow \pi_1(K/K^\sigma_{ss}) \longrightarrow \pi_0(K^\sigma_{ss}) \longrightarrow \cdots . \tag{4.1}$$

However, by ([2], p.223),

$$\pi_1(K) \;\cong\; \ker(\exp : \mathfrak{t} \to T)/\mathbf{Z}(\text{roots of } \mathfrak{k}).$$

Therefore, since K is semi-simple, $\pi_1(K)$ is finite. By compactness of K^σ_{ss}, $\pi_0(K^\sigma_{ss})$ is finite. Hence $\pi_1(K/K^\sigma_{ss})$, being caught in the middle in (4.1), is also finite. It follows that

$$H^1(K/K^\sigma_{ss}, \mathbf{C}) \cong Hom(\pi_1(K/K^\sigma_{ss}), \mathbf{C}) = 0,$$

which proves the lemma. □

We return to our pre-quantum line bundle $\mathbf{L}$ on $G/(P,P)$, corresponding to the $K \times T_\sigma$-invariant Kaehler structure ω. Let β be the 1-form $-\sqrt{-1}\partial F$, so $d\beta = \omega$. We claim that

Proposition 4.2. *There exists a non-vanishing section s_o on $\mathbf{L}$, with the property*

$$\beta = \frac{1}{\sqrt{-1}} \frac{\nabla s_o}{s_o}. \tag{4.2}$$

This section is unique up to a non-zero constant multiple, and is holomorphic. Up to a non-zero constant,

$$\langle s_o, s_o \rangle = e^{-F}.$$

Proof. Since $[\omega] = 0$, $\mathbf{L}$ is a trivial bundle; so there exists a nowhere zero section s_1 of $\mathbf{L}$. Let

$$\alpha = \frac{1}{\sqrt{-1}} \frac{\nabla s_1}{s_1}.$$

By the definition of the curvature form on $\mathbf{L}$, $d\alpha = \omega$; so $d(\beta - \alpha) = 0$. Since $H^1(G/(P,P), \mathbf{C}) = 0$, there exists a complex-valued function f such that $\beta = \alpha + df$. Let $s_o = (\exp \sqrt{-1} f) s_1$. Then

$$\frac{1}{\sqrt{-1}} \frac{\nabla s_o}{s_o} = \frac{1}{\sqrt{-1}} \frac{\nabla s_1}{s_1} + df = \beta.$$

This proves the existence of a holomorphic section s_o satisfying (4.2).

Suppose that s_1 and s_2 are two sections satisfying this formula. Let $h = \frac{s_2}{s_1}$. Then

$$\frac{1}{\sqrt{-1}} \frac{\nabla s_2}{s_2} = \frac{1}{\sqrt{-1}} \frac{\nabla s_1}{s_1} + \frac{1}{\sqrt{-1}} d \log h,$$

which implies that h is a constant. Hence, up to a constant, the solution of (4.2) is unique.

If v is an anti-holomorphic vector field, then

$$\frac{1}{\sqrt{-1}} \frac{\nabla_v s_o}{s_o} = \iota(v)\beta = 0,$$

as β is a form of type $(1,0)$. Hence s_o is holomorphic. Since β is $K \times T_\sigma$-invariant, s_o induces a $K \times T_\sigma$-representation on the space of holomorphic sections on $\mathbf{L}$, where s_o is $K \times T_\sigma$-invariant. Namely, given a holomorphic section $f s_o$ of $\mathbf{L}$ (note that s_o is non-vanishing), $K \times T_\sigma$ acts by

$$L_k^* R_t^* (f s_o) \;=\; (L_k^* R_t^* f) s_o, \quad k \in K, t \in T_\sigma, \tag{4.3}$$

where $L_k^* R_t^* f$ denotes the standard action on the holomorphic functions lifted from the $K \times T_\sigma$-action on $G/(P,P)$. Hence s_o defines a $K \times T_\sigma$-equivariant trivialization.

For this section s_o, we now show that $\langle s_o, s_o \rangle = e^{-F}$. By K-invariance, it suffices to show that this is the case when restricted to A_σ. Let σ be the cell corresponding to the parabolic subgroup P, and let m be the dimension of σ. We write $T_\sigma A_\sigma = \mathbf{C}^m/\mathbf{Z}^m = \{z_1, ..., z_m\}$ as in (3.1), so that F, being T_σ-invariant, is a function of y only. Let $\imath : T_\sigma A_\sigma \longrightarrow G/(P,P)$ be the natural inclusion. Then

$$\imath^*\beta = -\sqrt{-1}\partial F = \frac{1}{2}\sum_1^m \frac{\partial F}{\partial y_i} dz_i. \tag{4.4}$$

Let $\nabla_i = \nabla_{\partial/\partial y_i}$. Then

$$\frac{\partial}{\partial y_i}\langle s_o, s_o\rangle = \langle \nabla_i s_o, s_o\rangle + \langle s_o, \nabla_i s_o\rangle.$$

However, by (4.2) and (4.4),

$$\frac{\nabla_i s_o}{s_o} = \sqrt{-1}(\beta, \frac{\partial}{\partial y_i}) = -\frac{1}{2}\frac{\partial F}{\partial y_i}$$

so

$$\frac{\partial}{\partial y_i}\log\langle s_o, s_o\rangle = -\frac{\partial F}{\partial y_i}.$$

Therefore, up to a non-zero constant multiple,

$$\langle s_o, s_o\rangle = e^{-F}.$$

This proves the proposition. □

Let $\mathcal{O}(\mathbf{L})$ denote the space of holomorphic sections of the line bundle $\mathbf{L}$ on $G/(P,P)$. By Proposition 4.2, s_o induces a $K \times T_\sigma$-representation on $\mathcal{O}(\mathbf{L})$, given by (4.3), where s_o is $K \times T_\sigma$-invariant. Let $\lambda \in \mathfrak{t}_\sigma^*$ be a dominant integral weight, and let $\mathcal{O}(\mathbf{L})_\lambda$ denote the holomorphic sections that transform by λ under the right T_σ-action. Since this action commutes with the left K action, $\mathcal{O}(\mathbf{L})_\lambda$ is a K-subrepresentation of $\mathcal{O}(\mathbf{L})$. We now show that the K-finite vectors in $\mathcal{O}(\mathbf{L})$ decompose into $\{\mathcal{O}(\mathbf{L})_\lambda\ ;\ \lambda \in \overline{\sigma}\}$ as irreducible K-representations with highest weights λ. Using the holomorphic section s_o of Proposition 4.2, it suffices to consider the holomorphic functions $\mathcal{O}(G/(P,P))$, since

$$\mathcal{O}(G/(P,P)) \otimes s_o = \mathcal{O}(\mathbf{L})$$

is a $K \times T_\sigma$-equivariant trivialization.

Recall that $\overline{W}$ is the closure of the Weyl chamber W, and $\sigma \subset \overline{W}$ is the cell corresponding to P. Let $\overline{\sigma}$ denote its closure in $\overline{W}$. For a dominant integral weight $\lambda \in \mathfrak{t}^*$, let $\mathcal{O}_\lambda \subset \mathcal{O}(G/(P,P))$ denote the holomorphic functions that transform by λ under the right T_σ-action. Since the right T_σ-action commutes with the left K-action, each $\mathcal{O}_\lambda$ is a K-representation space.

Proposition 4.3. *The irreducible K-representation with highest weight λ occurs in $\mathcal{O}(G/(P,P))$ if and only if $\lambda \in \overline{\sigma}$. For $\lambda \in \overline{\sigma}$ it occurs with multiplicity one, and is given by $\mathcal{O}_\lambda$.*

Proof. The fibration π of (2.4) induces an injection of holomorphic functions,

$$\pi^* : \mathcal{O}(G/(P,P)) \longrightarrow \mathcal{O}(G/N).$$

This map intertwines with the $K \times T_\sigma$-action.

Let λ be a dominant integral weight, but suppose that $\lambda \notin \overline{\sigma}$. We shall show that the K-irreducible with highest weight λ does not occur in $\mathcal{O}(G/(P,P))$. By the Borel-Weil theorem, the K-irreducible with highest weight λ occurs in $\mathcal{O}(G/N)$ with multiplicity one, and can be taken as the holomorphic functions in G/N that transform by λ under the right T-action. We denote this space by $V_\lambda \subset \mathcal{O}(G/N)$. Since π^* is injective, it suffices to show that

$$\pi^*\mathcal{O}(G/(P,P)) \cap V_\lambda = 0. \tag{4.5}$$

Since $\lambda \notin \overline{\sigma}$, $(\lambda, \xi) \neq 0$ for some $\xi \in \mathfrak{t}_\sigma^\perp$. Let $0 \neq f \in V_\lambda$. Then the right action R_ξ^* on V_λ satisfies

$$R_\xi^* f = (\lambda, \xi) f \neq 0. \tag{4.6}$$

Since $T_\sigma^\perp$ is in the fiber of π, the image of π^* is $T_\sigma^\perp$-invariant. Therefore, (4.6) says that f cannot be in the image of π^*. This proves (4.5).

Conversely, suppose that $\lambda \in \overline{\sigma}$ is a dominant integral weight. We again let $V_\lambda \subset \mathcal{O}(G/N)$ be the holomorphic functions that transform by λ. By the Borel-Weil theorem, V_λ is an irreducible representation with highest weight λ, and such an irreducible occurs with multiplicity one. Therefore, to complete the proof of Proposition 4.3, we need to show that

$$V_\lambda \subset \pi^*\mathcal{O}(G/(P,P)), \quad \lambda \in \overline{\sigma}. \tag{4.7}$$

Recall from (2.2) that the fiber of π is $(K_{ss}^\sigma)_{\mathbf{C}}/(M \cap N) = K_{ss}^\sigma \times A_\sigma^\perp$. Choose a fiber of π, and let

$$\imath : K_{ss}^\sigma A_\sigma^\perp \hookrightarrow G/N$$

be a holomorphic $K_{ss}^\sigma \times A_\sigma^\perp$-equivariant imbedding as this fiber of π. Let $f \in V_\lambda$. We claim that f is constant on this fiber:

By applying the Borel-Weil theorem on $(K_{ss}^\sigma)_{\mathbf{C}}/(M \cap N) = K_{ss}^\sigma \times A_\sigma^\perp$, we see that $\imath^* f$, which is right $T_\sigma^\perp$-invariant since $(\lambda, \mathfrak{t}_\sigma^\perp) = 0$, has to be a constant function. Hence f is constant on that fiber, as claimed.

Since our argument is independent of the choice of fiber and element of V_λ, we conclude that every element of V_λ is constant on every fiber of π. This implies (4.7), and Proposition 4.3 is now proved. □

We have shown that the irreducible K-representation with highest weight λ occurs in $\mathcal{O}(\mathbf{L})$ if and only if $\lambda \in \overline{\sigma}$. For $\lambda \in \overline{\sigma}$, it occurs with multiplicity one, and is given by $\mathcal{O}(\mathbf{L})_\lambda$. We shall decide which of these irreducible K-representations are square-integrable, in the following sense.

From the description $G/(P,P) = (K/K_{ss}^\sigma) \times A_\sigma$, we see that there is a $K \times A_\sigma$-invariant measure μ on $G/(P,P)$, which is unique up to a non-zero constant. Given a holomorphic section s of $\mathbf{L}$, we consider the integral

$$\int_{G/(P,P)} \langle s, s \rangle \mu \, .$$

Let $H_\omega \subset \mathcal{O}(\mathbf{L})$ be the holomorphic sections in which this integral converges. Since the Hermitian structure $\langle , \rangle$ and μ are K-invariant, H_ω becomes a unitary K-representation space. The next proposition shows which irreducible K-representations occur in H_ω.

Let $\lambda \in \overline{\sigma}$ be a dominant integral weight. Let

$$\Phi : G/(P,P) \longrightarrow \mathfrak{k}^*$$

be the moment map of the K-action on $(G/(P,P), \omega)$. Recall that $\mathcal{O}_\lambda$ and $\mathcal{O}(\mathbf{L})_\lambda$ are respectively the holomorphic functions and sections that transform by $\lambda \in \mathfrak{t}_\sigma^*$ under the right T_σ-action.

Proposition 4.4. *Let $s \in \mathcal{O}(\mathbf{L})_\lambda$. Then $s \in H_\omega$ if and only if λ is in the image of the moment map.*

Proof. Let s_o be the unique holomorphic section of Proposition 4.2. Therefore, $\langle s_o, s_o \rangle = e^{-F}$, where F is the potential function of ω. Since s_o is non-vanishing and $K \times T_\sigma$-invariant,

$$\mathcal{O}(\mathbf{L})_\lambda = \mathcal{O}_\lambda \otimes s_o.$$

Therefore, we are reduced to showing that $f \in \mathcal{O}_\lambda$ satisfies

$$\int_{G/(P,P)} |f|^2 e^{-F} \mu \quad < \quad \infty \tag{4.8}$$

if and only if λ is in the image of Φ.

Here μ is the product of a K-invariant measure dk on K/K_{ss}^σ and a Haar measure da on A_σ. By the exponential map, the measure da on A_σ can be identified with the Lebesgue measure dy on $\mathbf{R}^m$, where $m = \dim \sigma$. Given $k \in K$, the left K-action on $\mathcal{O}_\lambda$, $L_k^* : \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\lambda$, is

$$(L_k^* f)(p) = f(kp).$$

Let $f_1, ..., f_N$ be a basis of $\mathcal{O}_\lambda$ which is orthonormal with respect to the (unique) K-invariant inner product on $\mathcal{O}_\lambda$. Given an element $f = \sum c_i f_i$ of $\mathcal{O}_\lambda$,

$$f(ky) = (L_k^* f)(y) = \sum c_i a_{ir}(k) f_r(y),$$

where $a_{ir}(k)$ is the irth matrix coefficient of the K-representation on $\mathcal{O}_\lambda$ with respect to the basis above. Thus

$$\int |f(ky)|^2 dk = \sum c_i \overline{c_j} (\int a_{ir}(k) \overline{a_{js}}(k) dk) f_r(y) \overline{f_s(y)},$$

where the integrals are taken over K/K_{ss}^σ. However, by Peter-Weyl the inner integral is equal to

$$\frac{1}{N} \delta_{ij} \delta_{rs}$$

([3], p.186), so the integral (4.8) reduces to

$$\frac{1}{N} \|f\|^2 \int_{\mathbf{R}^m} \sum |f_r(y)|^2 e^{-F(y)} dy, \tag{4.9}$$

where $\|f\|$ is the norm of f with respect to the given K-invariant inner product structure on $\mathcal{O}_\lambda$. However, each of the functions $f_r(y)$ transforms under the infinitesimal $\mathfrak{t}_\sigma$-action according to the character $\lambda \in \mathfrak{t}_\sigma^*$, and therefore, being holomorphic, transforms under the action of $\mathfrak{h}_\sigma = (\mathfrak{t}_\sigma)_{\mathbf{C}}$ according to the complexified character $\lambda_{\mathbf{C}} \in \mathfrak{h}_\sigma^*$. In particular, $|f_r(y)|^2$ is a constant multiple of $e^{2\lambda(y)}$. Hence if $f \neq 0$, (4.9) is a constant multiple of the integral

$$\int_{\mathbf{R}^m} e^{-F(y) + 2\lambda(y)} dy.$$

However, this integral converges if and only if 2λ is in the image of the Legendre transform of F ([4], Appendix); or equivalently if and only if λ is in the image of the moment map. This proves the proposition. $\square$

With this result, Theorem III follows. We see from Theorems II, III that not all irreducibles are contained in H_ω: The irreducible representation $\mathcal{O}(\mathbf{L})_\lambda$ with highest weight λ satisfies $\mathcal{O}(\mathbf{L})_\lambda \subset H_\omega$ if and only if $\lambda \in \frac{1}{2}L_F(\mathfrak{a}_\sigma) \subset \sigma$. This necessarily excludes $\lambda \in \overline{\sigma}\backslash\sigma$. However, in the next section, we shall see that the potential function F can be constructed such that $\frac{1}{2}L_F(\mathfrak{a}_\sigma) = \sigma$, and hence $\mathcal{O}(\mathbf{L})_\lambda \subset H_\omega$ for all $\lambda \in \sigma$.

5. Construction of a model

Let P be a parabolic subgroup of G, and σ its corresponding cell of dimension m, given in (1.2). There exist dominant fundamental weights $\alpha_1, ..., \alpha_m \in \mathfrak{a}_\sigma^*$ ([9], p. 498) such that

$$\sigma = \{\sum_1^m y_i\alpha_i \ ; \ y_i > 0\}.$$

Let $F_P : \mathfrak{a}_\sigma \longrightarrow \mathbf{R}$ be defined by

$$F_P(v) = \sum_1^m e^{\alpha_i(v)}. \tag{5.1}$$

Then $F_P \in C^\infty(\mathfrak{a}_\sigma)$ is strictly convex, and the image of its Legendre transform is exactly σ. Therefore, the moment map Φ satisfies $\Phi(A_\sigma) = \sigma$. Extend F_P to $G/(P,P)$ by K-invariance, and it follows from Theorem II that

$$\omega_P = \sqrt{-1}\partial\bar{\partial}F_P$$

is a Kaehler structure on $G/(P,P)$. Let $\mathbf{L}_P$ be the corresponding line bundle, described before. For a dominant integral weight λ, we let $\mathcal{O}(\mathbf{L}_P)_\lambda$ denote the holomorphic sections of $\mathbf{L}_P$ that transform by λ under the right T_σ-action. Let H_{ω_P} be the holomorphic sections that are square-integrable under (1.4), so that it is a unitary K-representation space. By Theorem III, $\mathcal{O}(\mathbf{L}_P)_\lambda$ is an irreducible K-representation with highest weight λ, whenever $\lambda \in \overline{\sigma}$. Further, since $\Phi(A_\sigma) = \sigma$, $\mathcal{O}(\mathbf{L}_P)_\lambda \subset H_{\omega_P}$ whenever $\lambda \in \sigma$.

We repeat this geometric construction among all the parabolic subgroups P containing the fixed Borel subgroup $B = HN$. In each case, we use F_P in (5.1) as the potential function for the Kaehler structure ω_P on $G/(P,P)$. Then the direct sum

$$\bigoplus_{B\subset P} H_{\omega_P}$$

is a model in the sense of I.M. Gelfand [7]: a unitary K-representation where all irreducibles occur with multiplicity one.

References

1. R. Abraham and J. Marsden, *Foundations of Mechanics, 2nd. ed.,* Addison-Wesley, 1985. MR **81e:**58025
2. T. Brocker and T. tom Dieck, *Representations of compact Lie groups,* Springer-Verlag, N.Y. 1985. MR **86i:**22023
3. C. Chevalley, *Theory of Lie Groups,* Princeton U. Press, Princeton 1946. MR **7:**412c
4. M.K. Chuah and V. Guillemin, *Kaehler structures on* $K_{\mathbf{C}}/N$, Contemporary Math. 154 : The Penrose transform and analytic cohomology in representation theory (1993), 181–195. MR **94k:**22028

5. V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations,* Invent. Math. 67 (1982), 515–538. MR **83m:**58040
6. V. Guillemin and S. Sternberg, *Symplectic techniques in physics,* Cambridge U. Press, Cambridge 1990. MR **91d:**58073
7. I.M. Gelfand and A. Zelevinski, *Models of representations of classical groups and their hidden symmetries,* Funct. Anal. Appl. 18 (1984), 183–198. MR **86i:**22024
8. S. Helgason, *Differential Geometry, Lie groups, and symmetric spaces,* Academic Press, 1978. MR **80k:**53081
9. S. Helgason, *Groups and Geometric Analysis,* Academic Press, 1984. MR **86c:**22017
10. A. Knapp, *Representation Theory of Semisimple Groups,* Princeton U. Press, Princeton 1986. MR **87j:**22022
11. B. Kostant, *Quantization and unitary representations,* Lecture Notes in Math. 170, Springer 1970, 87–208. MR **45:**3638
12. H.S. La, P. Nelson, A.S. Schwarz, *Virasoro Model Space,* Comm. Math. Phys. 134 (1990), 539–554. MR **92c:**22041

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